

Kalman Filtering With Model Uncertainties

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ABSTRACT

In the classical Kalman filter theory, one of the key assumptions is that a priori knowledge of the system model, which represents the actual system, is known without uncertainty. Our focus in this research is to estimate the state of a system that is subjected to stochastic disturbances by using an erroneous model along with the available stored measurements. We examine two approaches that take the effects of uncertain parameters into the account since these uncertain parameters degrade the estimate of the state. In the first approach, the errors in the nominal model, which are approximated by fictitious noise and covariance of the fictitious noise, are computed by using stored data. It is premised that the norm of discrepancy between correlation functions of the measurements and their estimates from the nominal model is minimum. The second approach involves the identification of a Kalman filter model on the premise that the norm of discrepancy between the measurements and their estimates is minimum. This paper reviews the two approaches and illustrates their performances numerically.

1. Introduction

The basic idea in estimation theory is to obtain approximations of the true response by using information from a model and from any available measurements. The mathematical structure used to perform estimation is known as an observer. The optimal observer for linear systems subjected to broad band disturbances is the Kalman Filter (KF), [1]. In the classical Kalman filter theory, one of the key assumptions is that a priori knowledge of the system model, which represents the actual system, is known without uncertainty. In reality, due to the complexity in the systems, it is often impractical (and sometimes impossible) to model them exactly. Therefore, there is considerable uncertainty about the system model and the error-free model assumption of classical Kalman filtering is not realistic in applications. Methods for addressing the Kalman filtering with model uncertainty can be classified into two groups: (1) Robust Kalman Filtering (RKF), (2) Adaptive Kalman Filtering. The key idea in RKF is to design a filter such that a range of model parameters are taken into account. The formulation of RKF requires solving two Riccati equations that is computationally intensive and impracticable in systems of large model size, [2]. The adaptive Kalman filtering can be categorized into two subgroups. One is simultaneous estimation of the parameters and the state, which is applicable in two ways: (1) The bootstrap approach, (2) The combined state and parameter estimation approach. In the bootstrap approach, the estimation is carried out in two steps. In the first step the states are estimated with the assumed nominal values of the parameters. In the second step the parameters are calculated using the recent estimates of the state from step one in addition to measurements, [3]. In the combined state estimation approach, the unknown parameters are augmented to the state vector for their online identification. This idea was initially introduced by Kopp and Orford [4], who derived a recursive relationship for the updated estimates of the parameters and state as a function of measurement. Since the problem posed as nonlinear, nonlinear filtering techniques such as extended Kalman filter (EKF) are used to obtain the combined estimates of parameters and state. The other approach in adaptive Kalman filtering, instead of estimating the uncertain parameters themselves, includes the effect of the uncertain parameters in state estimation. In this approach, the model errors are approximated by fictitious noise and the covariance of the noise is tuned based on an analytical criteria. To the best of the writer's knowledge, this idea is first applied by Jazwinski [5] who determined the covariance fictitious noise so as to produce consistency between theoretical statistics of Kalman filter innovations and its

experimental estimates.

The objective of this study is to address the uncertainty issue in model that is used in Kalman filtering. We examine the feasibility and merit of two approaches that takes the effects of the uncertain parameters of the nominal model into account in state estimation. In the first approach, the system is approximated with an equivalent stochastic model and the problem is addressed in off-line conditions. The model errors are approximated by fictitious noise and the covariance of the fictitious noise is calculated using a correlation based approach that minimizes of the norm of discrepancy between correlations function of measurements and their estimates from the nominal model. The second approach approximates the system with an equivalent Kalman filter model and the filter gain is calculated using the data on the premise that the norm of measurement error of the filter is minimum.

The fictitious noise approach examined calculates the noise covariance matrices using correlations approaches. The two correlations approaches that have received most attention in the noise covariance estimation problem are based on: 1) correlations of the innovation sequence and 2) correlations of the output. In the innovations approach one begins by "guessing" a filter gain and then the approach calculates the noise covariance matrices from analysis of the resulting innovations. The correlations approaches to estimate the covariance matrices of process and measurement noise for Kalman Filtering from the measured data began soon after introduction of the filter. Perhaps the most widely quoted strategies to carry out the estimation of noise covariance matrices are due to Mehra [6] and the subsequent paper by Carew and Bellanger [7]. A noteworthy contribution from this early work is the contributions by Neethling and Young [8], who suggested some computational adjustments that could be used to improve accuracy. Recently, some other contributions to the Mehra's approach on the estimation of noise covariance matrices are presented. Odelson, Rajamani and Rawlings applied the suggestions of Neethling and Young's on Mehra's approach and used the vector operator solution for state error covariance Riccati equation of suboptimal filter, [9]. Akesson et al. extended their work for mutually correlated process and measurement noise case, [10]. Bulut, Vines-Cavanaugh and Bernal compared the performance of the output and innovations correlations approaches to estimate noise covariance matrices, [11].

The paper is organized as follows: the next section provides a brief summary of the KF particularized to a time invariant linear system with stationary disturbances (which is a condition we have implicitly assumed throughout the previous discussion). The following section reviews two stochastic error modeling approaches for state estimation and the paper concludes with a numerical example.

2. The Kalman Filter

Consider a time invariant linear system with unmeasured disturbances $w(t)$ and available measurements $y(t)$ that are linearly related to the state vector $x(t)$. The system has the following description in sampled time

$$x_{k+1} = Ax_k + Bw_k \quad (1)$$

$$y_k = Cx_k + v_k \quad (2)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times r}$ and $C \in \mathbb{R}^{m \times n}$ are the transition, input to state, and state to output matrices, $y_k \in \mathbb{R}^{m \times 1}$ is the measurement vector and $x_k \in \mathbb{R}^{n \times 1}$ is the state. The sequence $w_k \in \mathbb{R}^{r \times 1}$ is the disturbance known as the process noise and $v_k \in \mathbb{R}^{m \times 1}$ is the measurement noise. In the treatment here, it is assumed that these are mutually correlated Gaussian stationary white noise sequences with zero mean and known covariance matrices, namely

$$E(w_k) = 0 \quad (3)$$

$$E(v_k) = 0 \quad (4)$$

and

$$E(w_k w_j^T) = Q \delta_{kj} \quad (5)$$

$$E(v_k v_j^T) = R \delta_{kj} \quad (6)$$

$$E(w_k v_j^T) = S \delta_{kj} \quad (7)$$

where δ_{kj} denotes the Kronecker delta function, and $E(\cdot)$ denotes expectation. Q and R are covariance matrices of the process and measurement noise and S is cross-covariance between them. For the system in eqs.1 and 2 the KF estimate of the state can be computed from

$$\hat{x}_{k+1} = A\hat{x}_k + K(y_k - C\hat{x}_k) \quad (8)$$

where \hat{x}_k is the estimate of x_k and K is the (steady state) KF gain that can be expressed in a number of alternative ways, a popular one is

$$K = (APC^T + BS)(CPC^T + R)^{-1} \quad (9)$$

where P , the steady state covariance of the state error, is the solution of the Riccati equation

$$P = APA^T - (APC^T + BS)(CPC^T + R)^{-1}(APC^T + BS)^T + BQB^T \quad (10)$$

The KF provides an estimate of the state for which trace of is minimal. The difference between measured and estimated output, namely $e_k = y_k - C\hat{x}_k$ in Eq.8 is known as innovations sequence of the filter which is a white process. The filter is initialized as follows

$$\hat{x}_0 = E(x_0) \quad (11)$$

3. Stochastic Modeling of Uncertainty

We suppose that the uncertainty in the state estimate, in addition to the disturbances, derives from error in the matrices of the state space model. Specifically, we consider the situation given by

$$x_{k+1} = (A_n + \Delta A)x_k + (B_n + \Delta B)w_k \quad (12)$$

$$y_k = Cx_k + v_k \quad (13)$$

where A_n and B_n are nominal model matrices; ΔA and ΔB error matrices and we assume that the noise covariance and error matrices are unknown. Our objective is to obtain an estimate of the state x_k using the information of nominal model matrices and stored data of measurement sequence y_k .

3.1 Fictitious Noise Approach

An approximation of the state sequence of the system in Eq.12-13 can be obtained from an equivalent stochastic model, namely

$$\bar{x}_{k+1} = A_n \bar{x}_k + \bar{w}_k \quad (14)$$

$$\bar{y}_k = C \bar{x}_k + \bar{v}_k \quad (15)$$

Suppose that \bar{w}_k and \bar{v}_k are white noise sequences, with covariance matrices \bar{Q} and \bar{R} , respectively and \bar{S} is the cross-covariance between them. The equivalent disturbance \bar{w}_k can be obtained by comparing Eqs.12 and 14, that is

$$\bar{w}_k = \Delta A \bar{x}_k + B w_k \quad (16)$$

We note that since the state sequence x_k is not a white process and the white noise approximation of \bar{w}_k in Eq.16 is theoretically not correct. However, we examine the merit of using fictitious noise covariance matrices that make the output correlations of the actual system and equivalent model approximately equal. If the \bar{Q} and \bar{R} are known, then the KF can be applied to the equivalent stochastic model in Eqs.14-15 to obtain an estimate of the state. Since the actual system and equivalent model are stochastic systems, the outputs y_k and \bar{y}_k can be characterized with their correlations functions.

Therefore, one can calculate covariance of \bar{w}_k and \bar{v}_k using the correlations approaches on the premise that the norm of discrepancy between correlation functions of y_k and \bar{y}_k is minimum, namely minimizing the cost function

$$J = \|corr(y) - corr(\bar{y})\| \quad (17)$$

The drawback of output correlations approach is that the calculations of the \bar{Q} and \bar{R} matrices are performed in two steps and it does not allow to force positive definitiveness of the solution for these matrices. Moreover, the output correlations approach requires very long data to obtain accurate estimates of noise covariance matrices since the measurements are generally overly correlated, [11]. Another approach to this problem uses the correlations of innovations process. In this approach, the available measurements are filtered with an arbitrary gain and the correlations of resulting innovations are used. Suppose that the measurements y_k and \bar{y}_k are filtered thorough an arbitrary filter, in which we denote the gain as K_0 and resulting innovations processes for y_k and \bar{y}_k are denoted as e_k and \bar{e}_k respectively. In this case, the covariance of \bar{w}_k and \bar{v}_k are calculated on the premise that the norm of discrepancy between correlation functions of e_k and \bar{e}_k is minimum, namely minimizing the cost function

$$J = \|corr(e) - corr(\bar{e})\| \quad (18)$$

The mathematical formulations of this approach is presented in the following. We begin with the expression for the covariance function of the innovation process for any stable observer with gain K_0 . As initially shown by Mehra [6] this function is

$$L_j = C\bar{P}C^T + \bar{R} \quad j = 0 \quad (19)$$

$$L_j = C\bar{A}^j\bar{P}C^T + C\bar{A}^{j-1}B\bar{S} - C\bar{A}^{j-1}K_0\bar{R} \quad j > 0 \quad (20)$$

where \bar{P} the covariance of the state error in the steady state, is the solution of the Riccati equation

$$\bar{P} = \bar{A}\bar{P}\bar{A} + K_0\bar{R}K_0^T + B\bar{Q}B^T - K_0\bar{S}B^T - B\bar{S}^TK_0^T \quad (21)$$

and

$$\bar{A} = A_n - K_0C \quad (22)$$

Applying vec operator to both sides of the auto-correlation function of the innovations in Eqs.19-20 one obtains

$$vec(L_j) = (C \otimes C)vec(\bar{P}) + vec(\bar{R}) \quad j = 0 \quad (23)$$

$$vec(L_j) = (C\bar{A}^j \otimes C)vec(\bar{P}) + (B^T \otimes C\bar{A}^{j-1})vec(\bar{S}) - (I \otimes C\bar{A}^{j-1}K_0)vec(\bar{R}) \quad j > 0 \quad (24)$$

and applying vec operator to error covariance equation in Eq.21, one has

$$vec(\bar{P}) = [I - (\bar{A} \otimes \bar{A})]^{-1}[(K_0 \otimes K_0)vec(\bar{R}) + B \otimes Bvec(\bar{Q}) - (B \otimes K_0)vec(\bar{S}) - (K_0 \otimes B)vec(\bar{S}^T)] \quad (25)$$

Substituting Eq.25 into Eqs.23 and 24, and adding the terms related to \bar{S}^T to the terms related to \bar{S} and canceling \bar{S}^T , one finds

$$vec(L_j) = \begin{bmatrix} h_j^Q & h_j^S & h_j^R \end{bmatrix} \begin{bmatrix} vec(\bar{Q}) \\ vec(\bar{S}) \\ vec(\bar{R}) \end{bmatrix} \quad (26)$$

where

$$h_j^Q = (C \otimes C)[I - (\bar{A} \otimes \bar{A})]^{-1}(B \otimes B) \quad j = 0 \quad (27)$$

$$h_j^Q = (C \otimes \bar{C} \bar{A}^j) [I - (\bar{A} \otimes \bar{A})]^{-1} (B \otimes B) \quad j > 0 \quad (28)$$

$$h_j^S = -2I(C \otimes C) [I - (\bar{A} \otimes \bar{A})]^{-1} (B \otimes K_0) \quad j = 0 \quad (29)$$

$$h_j^S = (B^T \otimes \bar{C} \bar{A}^{j-1}) - 2I[(C \otimes \bar{C} \bar{A}^j) [I - (\bar{A} \otimes \bar{A})]^{-1} (B \otimes K_0)] \quad j > 0 \quad (30)$$

$$h_j^R = (C \otimes C) [I - (\bar{A} \otimes \bar{A})]^{-1} (K_0 \otimes K_0) + I \quad j = 0 \quad (31)$$

$$h_j^R = (C \otimes \bar{C} \bar{A}^j) [I - (\bar{A} \otimes \bar{A})]^{-1} (K_0 \otimes K_0) - (I \otimes \bar{C} \bar{A}^{j-1} K_0) \quad j > 0 \quad (32)$$

Listing explicitly the correlation functions in Eq.26 for lags $j = 1, 2, \dots, p$ and writing in matrix form one has

$$HX = L \quad (33)$$

where

$$H = \begin{bmatrix} h_0^Q & h_0^S & h_0^R \\ h_1^Q & h_1^S & h_1^R \\ h_2^Q & h_2^S & h_2^R \\ \vdots & \vdots & \vdots \\ h_p^Q & h_p^S & h_p^R \end{bmatrix}, L = \begin{bmatrix} \text{vec}(\mathbf{L}_0) \\ \text{vec}(\mathbf{L}_1) \\ \text{vec}(\mathbf{L}_2) \\ \vdots \\ \text{vec}(\mathbf{L}_p) \end{bmatrix}, X = \begin{bmatrix} \text{vec}(\bar{Q}) \\ \text{vec}(\bar{S}) \\ \text{vec}(\bar{R}) \end{bmatrix} \quad (34)$$

Estimates of \bar{Q} , \bar{S} and \bar{R} can be obtained from Eq.33. From its inspection, one finds that H has dimensions $m^2 px(r^2 + m^2 + mr)$. The sufficient condition for the uniqueness of the solution of Eq.33 is defined as follows in the general case; the number of unknown parameters in Q and S have to be smaller than the product of number of measurements and the state. The error in solving Eq.33 for X is entirely connected to the fact that the L is approximate since it is constructed from sample correlation functions of the innovations which are estimated from finite duration signals, namely

$$\hat{L}_j \stackrel{\text{def}}{=} E(e_k e_{k-j}^T) = \frac{1}{N-j} \sum_{k=1}^{N-j} e_k e_{k-j}^T \quad (35)$$

where N is the number of time steps. Substituting \hat{L} as the estimate of L , the solution of Eq.33 can be presented as in the following.

$$\text{Case \#1 } mn \geq (r^2 + mr) \quad (36)$$

In this case H is full rank and there exists a unique minimum norm solution for a weighting matrix I given in the following,

$$\hat{X} = (H^T H)^{-1} H^T \hat{L} \quad (37)$$

$$\text{Case \#2 } mn < (r^2 + mr) \quad (38)$$

In this case the matrix is rank deficient, and the size of null space of H can be calculated from $t = r^2 - mn$. The solution is written as follows,

$$\hat{X} = \hat{X}_0 + \text{null}(H)Y \quad (39)$$

where \hat{X}_0 is the minimum norm solution given in Eq.37 and $Y \in \mathbb{R}^{t \times 1}$ is an arbitrary vector. Therefore, we conclude Eq.33 has infinite solution when $mn < (r^2 + mr)$. Although a unique solution for Q and S does not exist when $mn < (r^2 + mr)$, any of the solution for Q , R and S from Eq.33, still gives the optimal Kalman gain. K can be

calculated using classical approach which involves solving Riccati equation given in Eq.10 for error covariance, P and obtaining K from Eq.9. Moreover, since the \bar{w}_k and \bar{v}_k are fictitious, the uniqueness of the solution of the least square problem is not a concern, as long as the positive definitiveness of the covariance matrices are provided. Therefore, the information of actual covariance matrices of w_k and v_k does not apply any condition to the solution. For instance, one can force the equivalent disturbances \bar{w}_k and measurement noise \bar{v}_k noise to be mutually correlated, namely, $\bar{S} \neq 0$, although the actual system has mutually uncorrelated w_k and v_k . We note that the innovations correlations approach allows to enforce the positive semi-definiteness when solving for \bar{Q} , \bar{R} and \bar{S} from Eq.33.

3.2 Equivalent Kalman Filter Approach

An approximation of the state sequence of the system in Eq.12-13 can be calculated using a Kalman filter model that is constructed from the nominal model and available measurements. Suppose that operating condition is off-line and consider an output form Kalman filter is given, namely

$$\hat{x}_{k+1} = (A_n - KC)\hat{x}_k + Ky_k \quad (40)$$

$$\hat{y}_k = C\hat{x}_k \quad (41)$$

where \hat{y}_k is the measurement predictions of the filter. The main idea here, which is initially described by Juang, Chen and Phan [12], is to calculate the filter gain K by minimizing the norm of the discrepancy between available measurement y_k and its estimate \hat{y}_k from the filter model is minimum, namely, minimizing the cost function

$$J = \|y - \hat{y}\| \quad (42)$$

As opposed to the fictitious noise approach, which involves solving a single least-squares problem, the equivalent Kalman Filter approach calculates the K in two steps. In the first step markov parameters of the auto-regressive model are calculated by solving a least square problem. In the second step the K is calculated using markov parameters of moving average model and solving another least square problem. For details the reader is referred to [12].

4. Numerical Experiment: Five-DOF Spring Mass System

In this numerical experiment we use the five-DOF spring mass system depicted in Fig.1 in order to examine the uncertainty modeling methods for Kalman filtering.

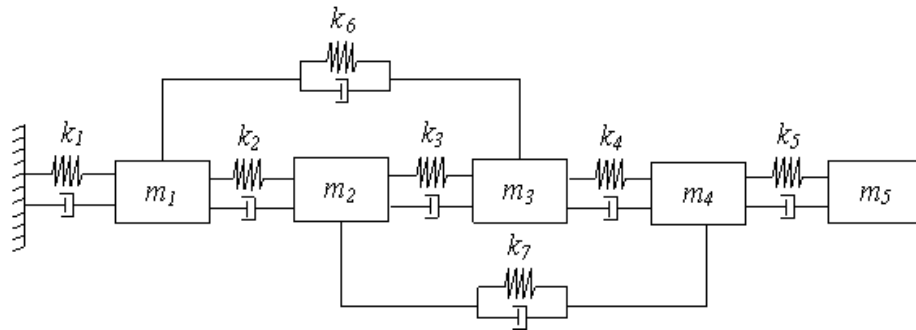


Fig. 1. Five-DOF spring mass system, $m_i = 0.05$, $k_i = 100$ (in consistent units). Damping is 2% in all modes.

We assume that true stiffness and mass values of the spring-mass system are given in consistent units as $k_i = 100$ and $m_i = 0.05$, respectively and take the spring stiffness values of the nominal model (A_n) as $\{80, 110, 90, 85, 110, 110, \text{ and } 105\}$. The un-damped frequencies of the system and the nominal model used in Kalman filtering are depicted in Table 1.

Table 1: The un-damped frequencies of the spring mass system and erroneous model.

Frequency No.	System	Model	% Change
1	0.582	0.545	6.349
2	1.591	1.594	0.184
3	2.851	2.883	1.104
4	3.183	3.119	2.008
5	3.434	3.470	1.073

We obtain results for output sensors at the third masses, which are recording velocity data at 100Hz sampling. The measurement noise is prescribed to have a root-mean-square (RMS) equal to approximately 10% of the RMS of the response measured. Unmeasured excitations and measurement noise are assumed to be mutually uncorrelated, with the covariance matrices,

$$Q = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \quad R = 0.030 \quad S = 0$$

Fictitious noise approach is applied using correlations of innovations process. The arbitrary filter gain K_0 , that is chosen such that eigenvalues of the matrix $(A_n - K_0 C)$ are assumed to have the same phase as those of A_n but with a 20% smaller radius. 80 lags of correlation functions of innovations process is taken into consideration and the sample innovation correlations functions are calculated using 600 seconds of data. The noise covariance matrices of the equivalent stochastic model is calculated as,

$$\bar{Q} = \begin{bmatrix} 11.86 & 5.64 & 4.33 & -7.23 & -0.015 \\ 5.64 & 3.58 & 0.90 & -1.80 & 1.33 \\ 4.33 & 0.90 & 3.27 & -4.99 & -1.94 \\ -7.23 & -1.80 & -4.99 & 7.69 & 2.719 \\ -0.015 & 1.33 & -1.94 & 2.719 & 2.244 \end{bmatrix} \quad \bar{R} = 0.0067 \quad \bar{S} = \begin{bmatrix} -0.218 \\ -0.058 \\ -0.147 \\ 0.227 \\ 0.077 \end{bmatrix}$$

The output correlation function of the actual system and equivalent stochastic model are depicted in Fig.1. The \bar{Q} , \bar{R} and \bar{S} are used to calculate a filter gain from the classical formulations of the Kalman filter and the state estimates are obtained from this filter.

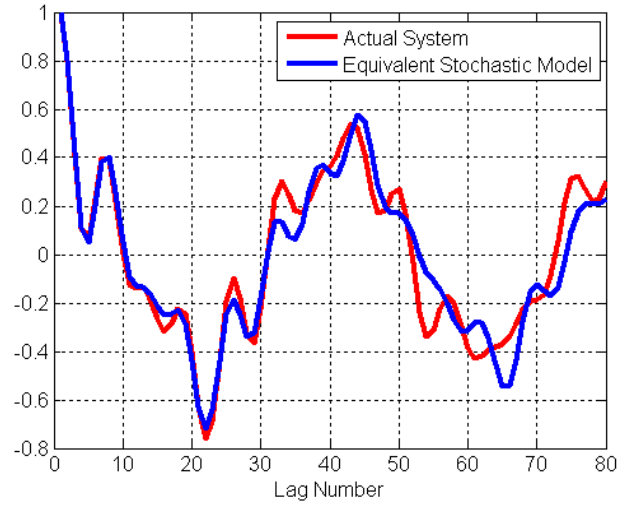


Fig. 2. The output correlation function of the five-DOF spring mass system.

Since the optimal estimate of the state can only be calculated using a Kalman filter that is constructed error free model of the actual system and the true noise covariance matrices, the methods examined in this study are suboptimal. For the comparison of the methods, we take the discrepancy between state estimate from optimal Kalman filter and suboptimal state estimate, namely

$$\varepsilon = \hat{x}_{optimal} - \hat{x}_{suboptimal} \quad (43)$$

and define the filter cost as

$$J = trace(E(\varepsilon\varepsilon^T)) \quad (44)$$

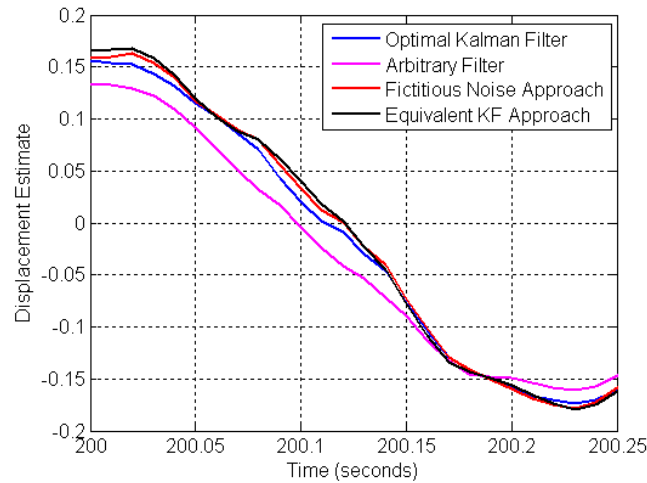


Fig. 3. Displacement estimate of the second mass

The displacement estimate of the second mass from the fictitious noise and equivalent Kalman filter approaches are depicted in Fig.3. As can be seen examined methods give better estimates compare to the arbitrary filter. Histograms of filter cost from 200 simulations for the examined methods and the arbitrary filter are depicted in Fig.4. As can be seen, the estimate from the

arbitrary filter is the worst with a mean of the filter cost $\mu = 11.84$. The fictitious noise approach performs better compare to the equivalent Kalman filter approach. The mean of the filter cost from 200 simulations are 1.60 and 2.65, respectively.

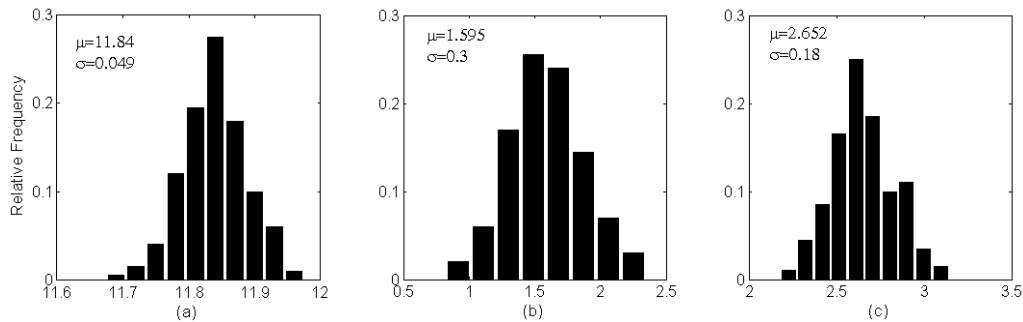


Fig. 4. Histograms of filter cost from 200 simulations, a) Arbitrary filter b) Fictitious Noise Approach c) Equivalent Kalman Filter Approach

5 Conclusions

This paper attempts to give a concise description of two stochastic error modeling approaches for state estimation using Kalman filter that is formulated using an erroneous model. The operating assumptions are that the system is linear time invariant and it is subjected to unmeasured Gaussian stationary disturbances and measurement noise, which are mutually correlated. In the first approach, model errors, are approximated by fictitious noise and covariance of the fictitious noise, are calculated using the data on the premise that the norm of discrepancy between correlation functions of the measurements and their estimates from the nominal model is minimum. The second approach examined approximates the system with an equivalent Kalman filter model. The filter gain is calculated using the data on the premise that the norm of measurement error of the filter is minimum. The two approaches are applicable in off-line conditions where stored measurement data is available. It leads to the expression that the fictitious noise approach is more complex than the equivalent Kalman filter approach, however the differences are not crucial when computer implementation are considered. Examination results show that although the state estimates from these two approaches are suboptimal, both approaches perform better than an arbitrary filter. In general, the fictitious noise approach performs better than the equivalent Kalman filter approach. However, the performance of the fictitious noise approach depends on the length of the data. This is due to the fact that the output correlations are calculated from a finite length sequences, and the accuracy of the sample correlations increases with longer data. Therefore, when the sample correlations are not calculated with a good approximation, for instance short data is considered, the equivalent Kalman filter approach gives better state estimates for the same data.

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