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Process and Measurement Noise Estimation for Kalman Filtering

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ABSTRACT

The Kalman Filter (KF) is the optimal observer for linear systems subjected to broadband disturbances. Its performance is assessed by computing the covariance of the state error, which requires the covariance matrices of the process and measurement noise Q and R. In practice, these matrices are seldom known a priori and it is necessary to approximate them from measurements. Among the developed approaches, the two that have received most attention are based on linear relations between these matrices and either: 1) the covariance function of the innovations from any stable filter or 2) the covariance function of the output measurements. This paper reviews the two approaches and implements them on a numerical example. The covariance of the state error is obtained and, through various cases of the noted example, it is shown how this variable is affected by signal duration, number of lags in the correlation functions, and the initial gain of the filter in the innovations approach.

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INTRODUCTION

The basic idea in estimation theory is to obtain closed-loop improved approximations of the true response. This can be achieved with an observer, also known as a state estimator, which uses the information encoded in discrepancies between observations and open loop model predictions. The Kalman Filter (KF) is the optimal observer for linear systems subjected to broad band disturbances. Mehra (1970) showed that it can be specified through an iterative process that uses a model of the system and only the covariance matrix of the measurement noise R. However, it is desirable to compute the covariance matrix of the process noise Q as well, as it is required to compute the covariance of the state error P; a means of assessing the KF's performance.

In practice, matrices Q and R are seldom known and one is faced with the question of how to specify them. Investigations into their approximation from measurements began soon after the KF was introduced and continue presently. This paper reviews two of the developed approaches: one based on the relationship of these matrices to the covariance function of the innovations from an arbitrary observer, and the other on the relation to the covariance function of the output measurements.

Heffes (1966) derived an expression for the covariance of the state error of any suboptimal filter as a function of Q and R. Mehra (1970) built off this expression to derive the innovations correlation approach for extracting these matrices. While this was a significant development, the approach did not offer a complete solution to the problem of estimating Q and R; specifically, inaccuracy existed due to unavoidable errors that lied in the estimation of the covariance of the innovations. Some modifications that could lead to improved performance were noted by Neethling and Young (1974), namely: 1) enforcement of symmetry 2) enforcement of semi-definitiveness in the covariance matrices and 3) formulation of the approach as an over-determined weighted least squares problem.

In addition to the noted modifications, performance is also affected by the selection of the initial gain. Mehra (1972) suggested that estimations of Q and R could be improved by repeating the computations using, as an initial gain, the filter gain obtained from the first iteration. This contention was found by Carew and Bellanger (1974) to be generally untrue. As these authors noted, if one starts with the exact Kalman gain in a first iteration, the results for Q and R, given the approximations due to finite duration, are such that the correct gain is not confirmed. Guidelines on how this initial gain should be selected for typical applications would be useful, but have yet to be put forth. Some observations on the topic are presented in this paper.

Regarding the output correlation approach, it was developed around the same time and is presented in a paper by Mehra (1972). It distinguishes itself by having simpler formulations and not requiring an initial gain. Furthermore, as noted by Mehra (1972), the approach has two additional limitations over the innovations approach, namely: that the output must be stable, and secondly, that because the output is more correlated than the innovations, this approach provides less efficient estimations of Q and R. As for similarities between the approaches, much like the errors that result in estimating the covariance of the innovations, the output measurements also contain error and lead to inaccuracy in the output correlation approach.

It is opportune to note that some recent work on the estimation of Q and R has taken place in the structural health monitoring community. The work appears to have been done in isolation of the noted classical works (Yuen et al. 2007), and accordingly, has influenced the motivation behind this paper. Thus, by offering a concise review of the correlation based approaches, this paper intends to bring attention to this important material for use in structural engineering applications. The rest of the paper is organized such that: the next section provides a brief summary of the KF particularized to a time invariant linear system with stationary disturbances; the following two sections review the formulations to estimate Q and R using the innovations and output correlations; and to conclude the paper, there is a numerical example and brief summary of the main points.

THE KALMAN FILTER

Consider a time invariant linear system with unmeasured disturbances w(t) and available measurements y(t) that are linearly related to the state vector x(t). The system has the following description in sampled time

$$x_{k+1} = Ax_k + Bw_k \tag{1}$$

$$y_k = Cx_k + v_k \tag{2}$$

where $A \in \mathbb{R}^{nxn}$, $B \in \mathbb{R}^{nxr}$ and $C \in \mathbb{R}^{mxn}$ are the transition, input to state and state to output matrices, $y_k \in \mathbb{R}^{mx1}$ is the measurement vector and $x_k \in \mathbb{R}^{nx1}$ is the state. The sequence $w_k \in \mathbb{R}^{nx1}$ is known as the process noise and $v_k \in \mathbb{R}^{mx1}$ is the measurement noise. In the treatment here, it is assumed that these are uncorrelated Gaussian stationary white noise sequences with zero mean and covariance of Q and R, namely

$$E(w_k) = 0 \quad E(w_k w_j^T) = Q\delta_{kj} \tag{3a,b}$$

$$E(v_k) = 0 \quad E(v_k v_j^T) = R\delta_{kj} \tag{4a,b}$$

$$E(v_k w_j^T) = 0 \tag{5}$$

where δ_{kj} denotes the Kronecker delta function and E(·) denotes expectation. For the system in eqs.1 and 2 the KF estimate of the state can be computed from

$$\hat{x}_{k+1}^{-} = A\hat{x}_k^{+} \tag{6}$$

$$\hat{x}_k^{+} = \hat{x}_k^{-} + K(y_k - C\hat{x}_k^{-}) \tag{7}$$

where $\hat{x}_k^{+}$ is the estimate after the information from the measurement at time k is taken into consideration and $\hat{x}_k^{-}$ is the estimate before. The (steady state) Kalman gain K can be expressed in a number of alternative ways, a popular one is (Simon 2006)

$$K = PC^T(CPC^T + R)^{-1} \tag{8}$$

where P, the steady state covariance of the state error, is the solution of the Riccati equation

$$P = A(P - PC^T(CPC^T + R)^{-1}CP)A^T + BQB^T \tag{9}$$

The KF provides an estimate of the state for which P is minimal. The filter is initialized as follows

$$\hat{x}_0^+ = E[x_0] \tag{10}$$

$$P_0 = E[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T] \tag{11}$$

INNOVATIONS CORRELATION APPROACH FOR Q AND R ESTIMATION

We begin with the expression for the covariance function of the innovation process for any stable observer with gain K_0. As shown by Mehra (1970) this function is

$$l_j = CPC^T + R \qquad j = 0 \tag{12}$$

$$l_j = C[A(I - K_0C)]^{j-1}A[PC^T - K_0(CPC^T + R)] \qquad j > 0 \tag{13}$$

where the covariance of the state error in the steady state, as shown by Heffes (1966), is the solution of the Riccati equation

$$P = A(I - K_0C)P(I - K_0C)^T A^T + AK_0RK_0{}^T A^T + BQB^T \tag{14}$$

<u>Estimation of PC^T</u>

Listing explicitly the covariance function for lags one to l one has

$$l_1 = CA(PC^T - K_0 l_0) \tag{15}$$

$$l_2 = C[A(I - K_0C)]A(PC^T - K_0 l_0) \tag{16}$$

$$l_3 = C[A(I - K_0C)]^2 A(PC^T - K_0 l_0) \tag{17}$$

....

$$l_l = C[A(I - K_0C)]^{l-1} A(PC^T - K_0 l_0) \tag{18}$$

from where one can write

$$L = Z(PC^T - K_0 l_0) \tag{19}$$

In which,

$$L = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ \ldots \\ l_l \end{bmatrix} \qquad Z = \begin{bmatrix} CA \\ C[A(I - K_0C)]A \\ C[A(I - K_0C)]^2 A \\ \ldots\ldots \\ C[A(I - K_0C)]^{l-1} A \end{bmatrix} \tag{20a,b}$$

As can be seen, matrix Z is the observability block of an observer whose gain is K_0 post-multiplied by the transition matrix A. On the assumption that the closed loop is stable and observable, one concludes that Z attains full column rank when l is no larger than the order of the system, *n*. Accepting that the matrix is full rank one finds that the unique least square solution to eq.19 is

$$PC^T = K_0 l_0 + Z^\dagger L \tag{21}$$

where $Z^\dagger$ is the pseudo-inverse of Z. On the premise that the model is known without error (which is a strong assumption), the error in solving eq.21 for PC^T results from the covariance of the innovations being an approximation from finite duration signals.

Estimation of R

Having obtained an estimate for PC^T, the covariance of the output noise R can be estimated from eq.12 as

$$\hat{R} = \hat{l}_0 - C(\hat{P}\hat{C}^T) \tag{22}$$

where the hats are added to emphasize that the quantities are estimates.

Estimation of Q

Derivation of an expression to estimate Q is significantly more involved than the one for R. The process begins by replacing, in eq.14, the covariance R by its expression in terms of the autocorrelation at zero lag. After some algebra one gets

$$P = APA^T + M + BQB^T \tag{23}$$

where

$$M = A[-K_0CP - PC^T K_0{}^T + K_0 l_0 K_0{}^T]A^T \tag{24}$$

Now consider a recursive solution for P in eq.23. In a first substitution one has

$$P = A(APA^T + M + BQB^T)A^T + M + BQB^T \tag{25}$$

or

$$P = A^2 P(A^2)^T + AMA^T + ABQB^T A^T + M + BQB^T \tag{26}$$

and after *q* substitutions one gets

$$P = A^q P(A^q)^T + \sum_{j=0}^{q-1} A^j M (A^j)^T + \sum_{j=0}^{q-1} A^j BQB^T (A^j)^T \tag{27}$$

Before solving eq.27 for Q it is first necessary to extract the set of equations for which only knowledge of PC^T (estimated by eq.21) is necessary. These equations are attained by postmultiplying both sides of eq.27 by C^T and pre-multiplying by CA^{-q}. This gives

$$CA^{-q}PC^T = CP(A^q)^T C^T + CA^{-q}\sum_{j=0}^{q-1} A^j M(A^j)^T C^T + CA^{-q}\sum_{j=0}^{q-1} A^j BQB^T(A^j)^T C^T \tag{28}$$

Given that P is symmetric, the CP product to the right of the equal sign can be expressed as $(PC^T)^T$, and one has

$$\sum_{j=0}^{q-1} CA^{j-q}BQB^T(A^j)^T C^T = CA^{-q}PC^T - (PC^T)^T(A^q)^T C^T - \sum_{j=0}^{q-1} CA^{j-q}M(A^j)^T C^T \tag{29}$$

For convenience we transpose both sides of eq.29 and get

$$\sum_{j=0}^{q-1} CA^j BQB^T(A^{j-q})^T C^T = (PC^T)^T(A^{-q})^T C^T - CA^q PC^T - \sum_{j=0}^{q-1} CA^j M(A^{j-q})^T C^T \tag{30}$$

where we've accounted for the fact that M is symmetrical. To shorten eq.30 we define

$$F_j = CA^j B \tag{31}$$

$$G_j = B^T(A^{j-q})^T C^T \tag{32}$$

and

$$s_q = (PC^T)^T(A^{-q})^T C^T - CA^q PC^T - \sum_{j=0}^{q-1} CA^j M \ (A^{j-q})^T C^T \tag{33}$$

So eq.30 becomes

$$\sum_{j=0}^{q-1} F_j Q G_j = s_q \tag{34}$$

Applying the *vec* operator to both sides of eq.34 one has

$$\sum_{j=0}^{q-1} (G_j^T \otimes F_j) \cdot vec(Q) = vec(s_q) \tag{35}$$

where $\otimes$ denotes the Kronecker product. Eq.35 can be evaluated for as many q values as one desires, although it is evident that all the equations obtained are not necessarily independent. Selecting q from one to p gives

$$H \cdot vec(Q) = S \tag{36}$$

where

$$h_q = \sum_{j=0}^{q-1} (G_j^T \otimes F_j) \quad H = \begin{bmatrix} h_1 \\ h_2 \\ .. \\ h_p \end{bmatrix} \quad S = \begin{bmatrix} vec(s_1) \\ vec(s_2) \\ .. \\ vec(s_p) \end{bmatrix} \tag{37a,b,c}$$

Eq.36 is the expression used in the innovations approach to obtain an estimate of Q. From its inspection one finds that H has dimensions m^2 p x r^2, where we recall that *m* and *r* represent the numbers of outputs and independent disturbances, respectively.

Structure in Q

The matrix Q is symmetrical and can often be assumed diagonal. The constraints of symmetry and/or a diagonal nature of Q can be reflected in a linear transformation of the form

$$vec(Q) = T \cdot vec(Q_r) \tag{38}$$

where $vec(Q_r)$ is the vector of unknown entries in Q after all the constraints are imposed. In the most general case, where only symmetry is imposed, the dimension of T is $n^2 \times r(r+1)/2$. However, since structural engineering problems are such that the largest possible value of r is n/2, T has a maximum dimension of $n^2 \times n(n+1)/4$. In the most general case in structures, therefore, the matrix H, after symmetry is imposed, is $m^2 n \times n(n+1)/4$. A necessary (albeit not sufficient) condition for a unique solution for Q, therefore, is $m \geq \sqrt{n+1}/2$.

Enforcing Positive Semi-Definitiveness

By definition, the true matrix Q is positive semi-definite (i.e., all its eigenvalues are ≥0), however, due to approximations, the least square solution may not satisfy this requirement. In the general case one can satisfy positive semi-definitiveness by recasting the problem as an optimization with constraints. Namely, minimize the norm of $(V \times vec(Q) - vec(G))$ subject to the constraint that all eigenvalues of Q ≥ 0. This problem is particularly simple for our case, where Q is diagonal. In this scenario the entries in *vec*(Q) are the eigenvalues and all that is required is to ensure that they are not negative. Also, here is a suitable application for the always converging non-negative least squares solution (Lawson and Hanson 1974). This algorithm is used for the numerical example provided at the end of this paper.

OUTPUT CORRELATION APPROACH FOR Q AND R ESTIMATION

As in the previous case, it is assumed that the state is stationary and that the process and measurement noise are white and uncorrelated with each other, namely

$$E(x_{k+1}x_{k+1}^T) = \Sigma \tag{39}$$

$$E(x_k v_k^T) = 0 \tag{40}$$

$$E(x_k w_k^T) = 0 \tag{41}$$

$$E(\omega_k v_k^T) = 0 \tag{42}$$

Estimation of R

The covariance function of the output is

$$\Lambda_i = E(y_{k+i} y_k^T) \tag{43}$$

Substituting eqs.2 and enforcing the assumptions in 40-42 one gets

$$\Lambda_i = C(E\left(x_{k+i} x_k^T\right))C^T \tag{44}$$

From eq.1 one can show that

$$x_{k+i} x_k^T = A^i x_k x_k^T + A^{i-1} B w_k x_k^T + A^{i-2} B w_{k+1} x_k^T + \ldots + B w_{k+i-1} x_k^T \tag{45}$$

Taking expectations on both sides of eq.45 gives

$$E(x_{k+i} x_k^T) = A^i E(x_k x_k^T) \tag{46}$$

which when substituted into eq.44 gives

$$\Lambda_0 = C \Sigma C^T + R \quad for \quad i = 0 \tag{47}$$

$$\Lambda_i = C A^i \Sigma C^T \quad for \quad i \neq 0 \tag{48}$$

Defining

$$G = E(x_{k+1} y_k^T) \tag{49}$$

and substituting eqs.1 and 2, then imposing the assumptions in eqs.40-42 one gets

$$G = A \Sigma C^T \tag{50}$$

Writing out the covariance function in eq.48 for i=1,2… *p* one has

$$\begin{Bmatrix} \Lambda_1 \\ \Lambda_2 \\ . \\ \Lambda_p \end{Bmatrix} = \begin{bmatrix} C \\ CA \\ . \\ CA^{p-1} \end{bmatrix} A \Sigma C^T \tag{51}$$

or, substituting Eq.50 into Eq.51 and solving for G one has

$$G = Ob_p^{\dagger} \cdot \begin{Bmatrix} \Lambda_1 \\ \Lambda_2 \\ . \\ \Lambda_p \end{Bmatrix} \tag{52}$$

From eq.47 and eq.50 one gets

$$R = \Lambda_0 - CA^{-1}G \tag{53}$$

Estimation of Q

Substituting eq.1 into eq.39 and enforcing the operating assumptions gives

$$\Sigma = A\Sigma A^T + BQB^T \tag{54}$$

Applying the *vec* operator to eqs.54 and eq.50 one obtains

$$vec(\Sigma) = \left(I - (A \otimes A)\right)^{-1} (B \otimes B) vec(Q) \tag{55}$$

and

$$vec(G) = \left(C \otimes A\right) vec(\Sigma) \tag{56}$$

Combining eqs.55 and 56 gives

$$vec(G) = V \cdot vec(Q) \tag{57}$$

where

$$V = \left(C \otimes A\right)\left(I - (A \otimes A)\right)^{-1} (B \otimes B) \tag{58}$$

Structure in Q

The observations made in the case of the innovations approach apply here also.

Enforcing Positive Semi-Definitiveness

By definition the true matrix Q is positive semi-definite (i.e., all its eigenvalues are ≥0) but the least square solution (due to approximations) may not lead to a solution that satisfies this requirement. In the general case one can satisfy positive semi-definitiveness by recasting the problem as an optimization with constraints. Namely: minimize the norm of $(V \times vec(Q) - vec(G))$ subject to the constraint that all eigenvalues of Q ≥ 0. The problem is particularly simple, however, in the case where Q is diagonal because the entries in *vec*(Q) are the eigenvalues and all that is required is to ensure that they are not negative. The non-negative least square solution, developed by Lawson and Hanson (1974), is guaranteed to converge and is used in the numerical simulations of the next section. The observations made on semi-definitiveness here apply equally in the case of the innovations correlation approach.

NUMERICAL EXAMINATION

We consider the 5-DOF spring-mass structure depicted in fig.1. The first un-damped frequency is 2.66Hz and the 5th is 16.30Hz. Damping is classical with 2% in each mode and the stochastic disturbance, having a Q = 1, is applied at the 1st coordinate. Velocity measurements are taken at the 3rd coordinate and the exact response is computed at 50Hz sampling. Noise in the output is such that R = $5x10^{-4}$. Two hundred simulations are performed on each of four different cases to investigate the affect of duration and number of lags on the estimation of Q and R. The cases are defined as follows: Case I: duration 200sec lags=40; Case II: duration 200sec. lags=10; Case III: duration 20sec. lags=40; and Case IV: duration 20sec. lags=10.

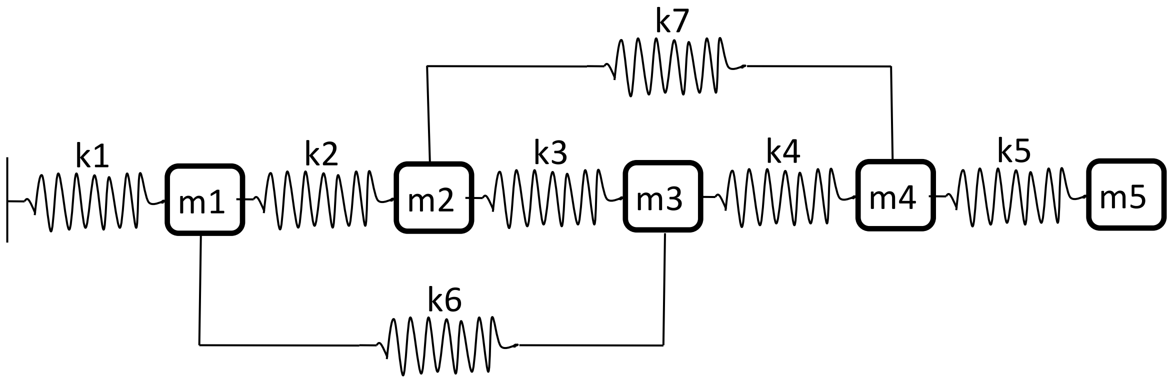

Fig.1 5-DOF spring-mass structure (m_1-m_5 = 0.05; k_1,k_3,k_5,k_7 =100; k_2,k_4,k_6=120)

The estimates of the Q and R obtained using presented methods and steady state error covariance matrices are calculated for each Q and R duplex from eq.9. The negative estimates of R are assumed to be zero because the eq.9 requires the Q and R have to be positive definite. Results are presented as scatter plots of R vs Q in figs. 2 and 3 and the histogram plots of state error covariance in figs. 4 and 5. The x-axis is the relative frequency in the figs 4 and 5, and the m, s and t are the mean, standard deviation and true value. P_{jj} is the j^{th} element of the diagonals.

The basic observations are as follows: In fig4 and fig5

1. In general the innovations approach performed better than the output correlation scheme.
2. Increasing the number of lags improved accuracy in the innovations approach but in the output covariance method the opposite result was obtained. Examination of why this is the case is currently ongoing.
3. As expected, the duration of the signals used in the calculations plays a critical role in the accuracy attained.

Influence of the Initial Gain

A question of interest is how selection of the initial gain in the innovations approach affects accuracy. Intuitively one expects that the closer the initial guess is to the true Kalman gain the more accurate the results will be. It appears from inspection, however, that this is not the case. Specifically, looking at eqs.20 and 21 one notes that the existence of very small singular values in Z will increase the vulnerability of PC^T to error on the empirical innovations. These singular values occur if some poles of the closed loop observer are small enough to become negligible for powers less than the system order; in the context of the Kalman filter, this occurs when the measurement noise is very small. To exemplify the previous observations, we consider the same system illustrated in fig.1, except that output measurements are assumed available not only at coordinate 3, but also at 5. Fig.6 shows histograms of Q based on 200 simulations for the case where the initial gains are: a) zero b) the exact KF and c) a filter based on Q=1, R=3*I. The duration of the signals is 40 secs and 10 lags are used. As can be seen, the result when the initial gain is the exact Kalman is the most precise, which is the result one intuitively anticipates. Fig.7, however, illustrates analogous results for the case where the measurement noise is decreased by 3 orders of magnitude. As can be seen, in this case (a) and (c) are (essentially) identical to those in fig.6 but the results are poor when the initial gain is the exact Kalman.

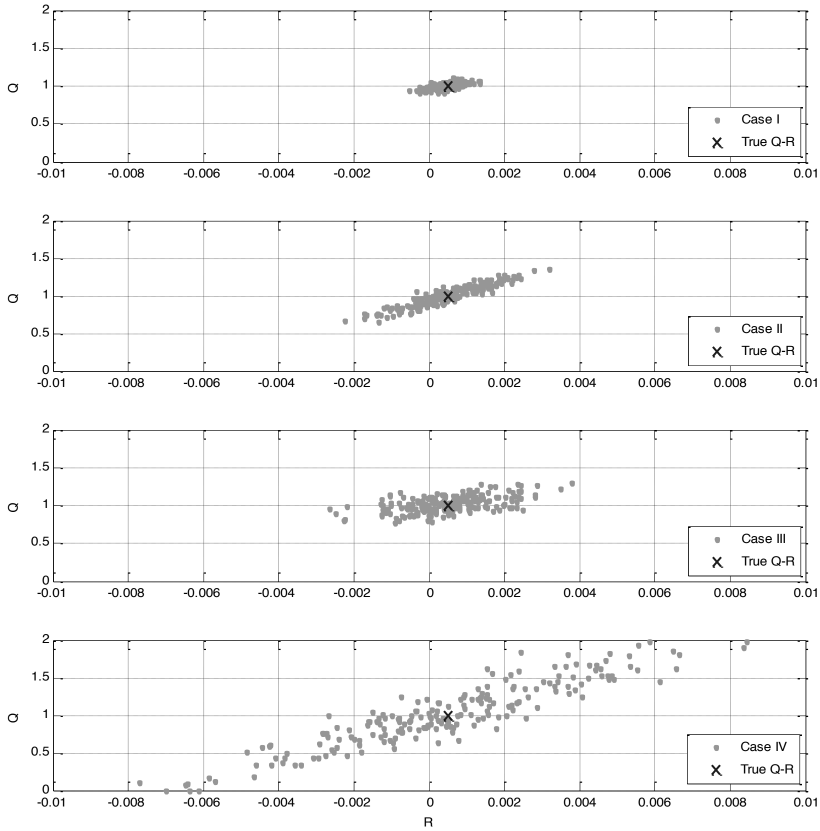

Fig.2 Innovations correlation approach Q-R estimations

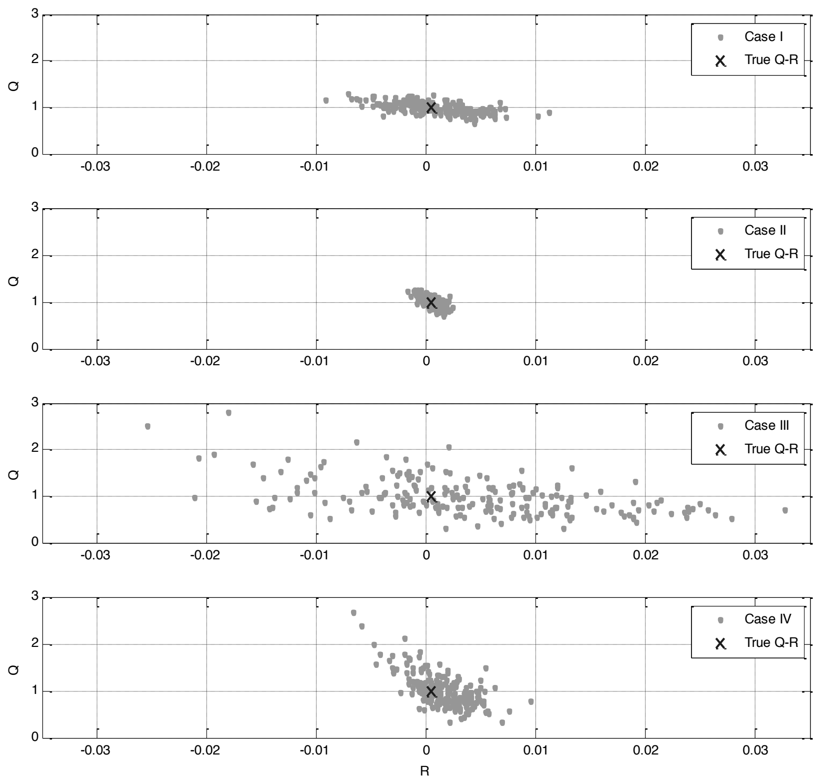

Fig.3 Output correlation approach Q-R estimations

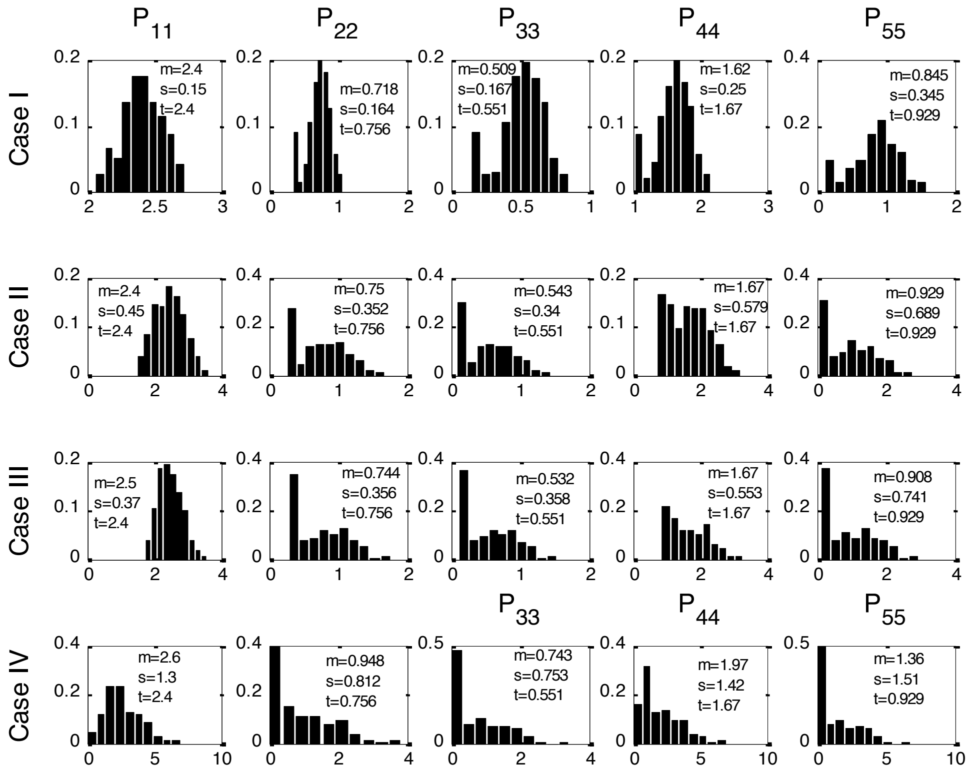

Fig.4 Histograms of the first five diagonal elements of the state error covariance obtained using the estimates of Q and R from innovations approach.

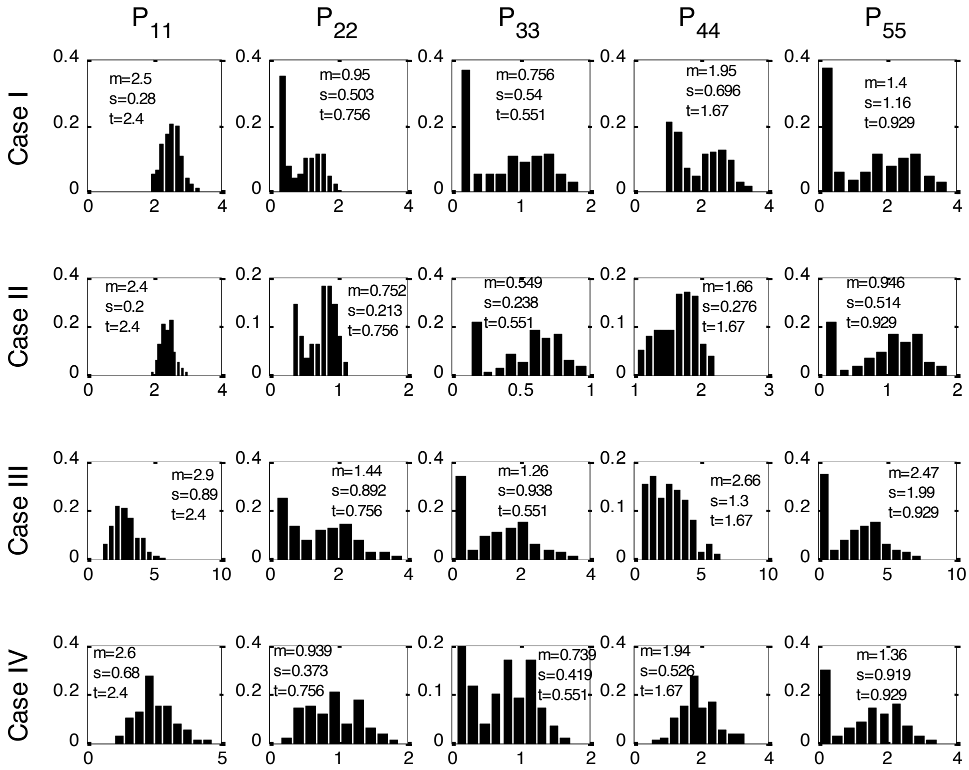

Fig.5 Histograms of the first five diagonal elements of the state error covariance obtained using the estimates of Q and R from output approach.

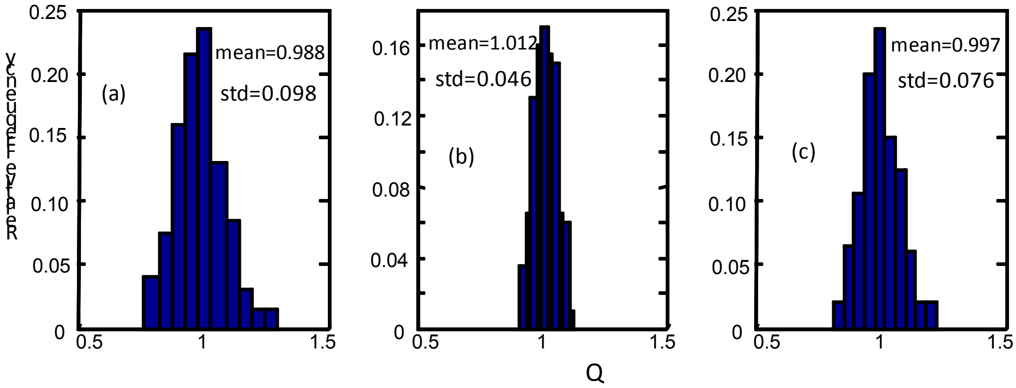

Fig.6 Histograms of Q with measurement noise R = 5e^{-4} x I: a) gain = 0, b) gain = exact KF and c) gain = KF for Q = 1, R = 3 x I

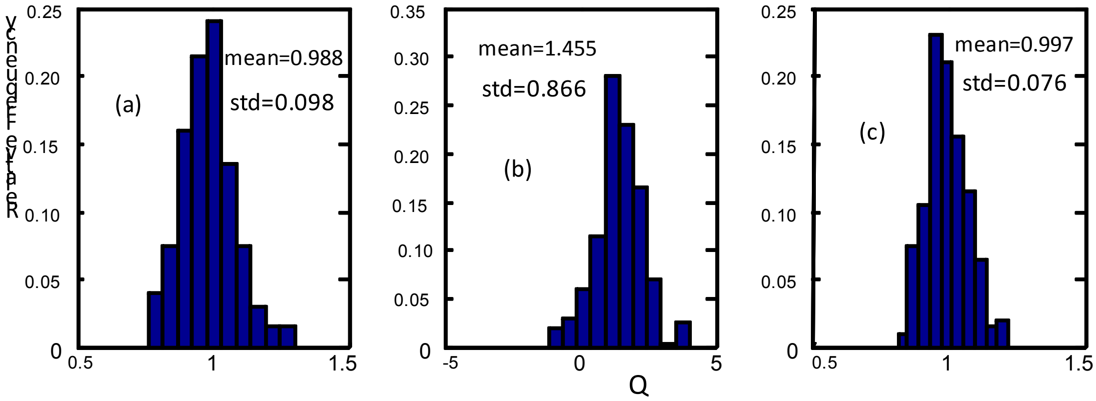

Fig.7 Histograms of Q with measurement noise R = 5e^{-7} x I: a) gain = 0, b) gain = exact KF and c) gain = KF for Q = 1, R = 3 x I

CONCLUDING COMMENTS

This paper attempts to give a concise description of the innovations and output correlation methods for estimating the covariance of the process and measurement noise Q and R. The operating assumptions are that the system is linear and time invariant and that the computations can be carried out offline. The innovations approach leads to expressions that are more complex than the output covariance scheme but the differences are not important when it comes down to computer implementation. It is shown that when selecting the initial gain in the innovations approach one should be careful not to create a model that has some small closed-loop poles. The reasoning behind this is that the small poles will cause the observability block (of the observer) to become poorly conditioned and result in a loss of accuracy in the estimation. Furthermore, it is shown that the foregoing consideration applies even in the case where the initial gain is the exact Kalman filter.

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